The Scenario Approach for Systems and Control Design ⋆

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Abstract: The 'scenario approach' is an innovative technology that has been introduced to solve convex optimization problems with an infinite number of constraints, a class of problems which often occurs when dealing with uncertainty. This technology relies on random sampling of constraints, and provides a powerful means for solving a variety of design problems in systems and control. The objective of this paper is to illustrate the scenario approach at a tutorial level, focusing mainly on algorithmic aspects. Its versatility and virtues will be pointed out through a number of examples in model reduction, robust and optimal control.

Keywords: Systems and control design; Robust convex optimization; Probabilistic methods; Randomized algorithms.

1. INTRODUCTION

Many problems in systems and control can be formulated as optimization problems, Boyd et al. [1994], Goodwin et al. [2005]. Here, we focus on optimization problems of *convex* type, Boyd and Vandenberghe [2004]. Convexity is appealing since 'convex' - as opposed to 'non-convex' - means 'solvable' in many cases. This observation has much influenced the systems and control community in recent years, as witnessed by an increasing interest in convex LMIs (Linear Matrix Inequalities) reformulations of a number of classical problems (Apkarian and Tuan [2000], Apkarian et al. [2001], Boyd et al. [1994], Gahinet [1996], Scherer [2005, 2006], Vandenberghe and Boyd [1996]), a process also fostered by the development of ever more effective convex optimization solvers (Boyd and Vandenberghe [2004], Grant et al. [2006, 2007]).

In practical problems, an often-encountered feature is that the environment is uncertain, i.e. some elements and/or variables are not known with precision. A common approach to counteract uncertainty is to robustify the design by considering a *min-max* optimization problem of the type

$$\min_{\xi} \max_{\delta \in \Delta} \ell_\delta(\xi), \tag{1}$$

where $\ell_\delta(\xi)$ (here assumed to be convex in ξ) represents the cost incurred when the design parameter value is ξ and the uncertainty parameter is δ. In this min-max approach, one tries to achieve the best performance which is guaranteed for all possible uncertainty instances in Δ.

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The min-max problem (1) is just a special case of a robust convex optimization program, Ben-Tal and Nemirovski [1998, 1999], El Ghaoui and Lebret [1997, 1998], where a linear objective is minimized subject to a number of convex constraints, one for each instance of the uncertainty:

$$\begin{aligned} \text{RCP} : \min_{\gamma \in \mathbb{R}^d} \; & c^T \gamma \\ \text{subject to: } & f_\delta(\gamma) \leq 0, \; \forall \delta \in \Delta, \end{aligned} \tag{2}$$

where $f_\delta(\gamma)$ are convex functions in the d-dimensional optimization variable γ for every δ within the uncertainty set Δ. Precisely, problem (1) can be re-written in form (2) as follows:

$$\begin{aligned} \min_{h,\xi} \; & h \\ \text{subject to: } & \ell_\delta(\xi) \leq h, \; \forall \delta \in \Delta, \end{aligned} \tag{3}$$

where $\gamma = (h, \xi)$, $c^T \gamma = h$, and $f_\delta(\gamma) = \ell_\delta(\xi) - h$ in this case. Note that, given a ξ, the slack variable h represents an upper bound to the cost $\ell_\delta(\xi)$ achieved by parameter ξ when δ ranges over the uncertainty set Δ. By solving (3) we seek that $\overline{\xi}$ that corresponds to the smallest upper bound $\overline{h}$.

More often than not, the uncertainty set Δ is a continuous set containing an infinite number of instances. Problems with a finite number of optimization variables and an infinite number of constraints are called *semi-infinite* optimization problems in the mathematical programming literature, Boyd and Vandenberghe [2004]. Reportedly, these problems are difficult to solve and are even NP-hard in many cases, Ben-Tal and Nemirovski [1998, 2002], Blondel and Tsitsiklis [2000], Braatz et al. [1994], Nemirovski [1993], Stengel and Ray [1991], Tempo et al. [2005], Vidyasagar [2001]. In other words, *though convex is easy, robust convex is difficult.*

In Calafiore and Campi [2005, 2006], an innovative technology called 'scenario approach' has been introduced to deal with semi-infinite convex programming at a very general level. The main thrust of this technology is that solvability can be obtained through random sampling of constraints provided that a probabilistic relaxation of the worst-case robust paradigm of (2) is accepted. Such probabilistic relaxation consists in being content with robustness against the *large majority* of the situations rather than against *all* situations. The good news is that in the scenario approach such large majority is under the control of the designer and can be made *arbitrarily close* to the set of all situations.

Random sampling has been used in a variety of fields besides optimization, and is e.g. at the basis of the truly vast literature on Monte-Carlo methods, Hammersley and Handscomb [1964], Fishman [1999], Robert and Casella [2004], Shapiro, Dentcheva and Ruszczynski [2009]. Along the scenario approach, random sampling has been brought into robust convex optimization resulting into a mathematically sound and practically useful algorithmic approach.

When dealing with problems in systems and control, the scenario approach permits to tackle situations where more standard approaches fail due to computational difficulties, and opens up new resolution avenues that get around traditional stumbling blocks in the design of devices incorporating robustness features.

The objective of the present paper is to introduce and illustrate at a tutorial level the scenario approach. The presentation will be user-oriented, with the main focus on algorithmic aspects, and will primarily consist of a number of examples in different contexts of systems and control to show the versatility of the approach.

Structure of the paper

After describing in Section 2 the scenario approach along with the concept of probabilistic relaxation of the RCP solution, we move to illustrating some possible applications of the approach to systems and control in Section 3. In particular, problems from robust control, optimal control with constraints, and model reduction are respectively treated in Sections 3.1, 3.2, and 3.3. Some final conclusions are drawn in Section 4.

2. THE SCENARIO APPROACH

The scenario approach presumes a probabilistic description of uncertainty, that is uncertainty is characterized through a set Δ describing the set of admissible situations, and a probability distribution Pr over Δ. Depending on the problem at hand, Pr can have different interpretations. Sometimes it is a measure of the likelihood with which situations occur, other times it simply describes the relative importance we attribute to different uncertainty instances. A probabilistic description of uncertainty is gaining increasing popularity within the control community as witnessed by many contributions such as Alamo et al. [2007, 2008], Barmish and Lagoa [1997], Calafiore and Campi [2006], Calafiore et al. [2000], Fujisaki et al. [2003], Khargonekar and Tikku [1996], Kanev et al. [2003], Lagoa [2003], Lagoa et al. [1998], Oishi and Kimura [2003], Polyak and Tempo [2001], Ray and Stengel [1993], Stengel and Ray [1991], Tempo et al. [1997, 2005], Vidyasagar [1997, 2001].

The scenario approach goes as follows. Since we are unable to deal with the wealth of constraints $f_\delta(\gamma) \leq 0$, $\forall \delta \in \Delta$, we concentrate attention on just a few of them by extracting at random N instances or 'scenarios' of the uncertainty parameter δ according to probability Pr. Only the constraints corresponding to the extracted δ's are considered in the scenario optimization:

SCENARIO OPTIMIZATION

Extract N independent identically distributed samples $\delta^{(1)}, \ldots, \delta^{(N)}$ according to probability Pr and solve the scenario convex program:

$$\begin{aligned} \mathrm{SCP}_N : & \min_{\gamma \in \mathbb{R}^d} c^T \gamma \\ & \text{subject to: } f_{\delta^{(i)}}(\gamma) \leq 0, \ i = 1, \ldots, N. \end{aligned}$$

Contrary to the RCP in (2), SCP_N is a standard convex *finite* (i.e. with a finite number of constraints) optimization problem and, consequently, a solution can be found at low computational cost via available solvers, provided that N is not too large. That is sampling has led us back to an easily solvable program.

Though totally disregarding all constraints but N of them may appear naive, the scenario approach stands on a very solid mathematical footing.
Since SCP_N is less constrained than RCP, its optimal solution γ_N^* is certainly super-optimal for RCP, that is $c^T\gamma_N^* \leq c^T\overline{\gamma}$, $\overline{\gamma}$ being the optimal RCP solution. On the other hand, an obvious question to ask is: what is the degree of robustness of γ_N^*, being this latter based on a finite number of constraints only? Precisely, what can we claim regarding the satisfaction or violation of all the other constraints, those we have not taken into consideration while optimizing? The following theorem, which is at the core of the scenario approach, shows that γ_N^* actually satisfies all unseen constraints except a user-chosen fraction that tends rapidly to zero as N increases.

Theorem 1. Select a 'violation parameter' $\epsilon \in (0, 1)$ and a 'confidence parameter' $\beta \in (0, 1)$.
If

$$N \geq \frac{2}{\epsilon}\left(\ln \frac{1}{\beta} + d\right) \tag{4}$$

(recall that d is the number of optimization variables), then, with probability no smaller than $1 - \beta$, γ_N^* satisfies all constrains in Δ but at most an ϵ-fraction, i.e. $Pr(\delta : f_\delta(\gamma_N^*) \not\leq 0) \leq \epsilon$. ⋆

Formula (4) provides an explicit expression for N as a function of ϵ and β and can be derived from the results in Campi and Garatti [2008]. Precisely, in Campi and Garatti [2008] the more general result is proven that Theorem 1 holds if N is chosen so as to satisfy condition

$$\sum_{i=0}^{d-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \beta. \tag{5}$$

Equation (4) is obtained from this result by making explicit (5) with respect to N, as done in Calafiore [2008]. Formula (5) is tight since the inequality in (5) becomes an equality for a whole class of problems, those called fully-supported in Campi and Garatti [2008].

Let us read through Theorem 1 in some detail. If we neglect the parts associated with β, then, the theorem simply says that the solution γ_N^* is robust against uncertainty in Δ up to a desired level ϵ. Moreover, ϵ can be made small at will by suitably choosing N. This means that, in the scenario approach, although the requirement to be robust against all situations is renounced, the right to decide which level of robustness is satisfactory is retained.

As for the probability $1-\beta$, one should note that γ_N^* is a random quantity because it depends on the randomly extracted constraints corresponding to $\delta^{(1)}, \ldots \delta^{(N)}$. It may happen that the extracted constraints are not representative enough of the other unseen constraints (one can even stumble on an extraction as bad as selecting N times the same constraint!). In this case no generalization can be expected, and the portion of unseen constraints violated by γ_N^* will be larger than ϵ. Parameter β controls the probability that this happens and the final result that γ_N^* violates at most an ϵ-fraction of constraints holds with probability $1-\beta$.
In theory, β plays an important role and selecting $\beta = 0$ yields $N = \infty$. For any practical purpose, however, β has very marginal importance since it appears in (4) under the sign of logarithm: we can select β to be such a small number as 10^{-10} or even 10^{-20}, in practice zero, and still N does not grow significantly.

To allow for a more immediate understanding, a pictorial representation of Theorem 1 is given in Figure 1.

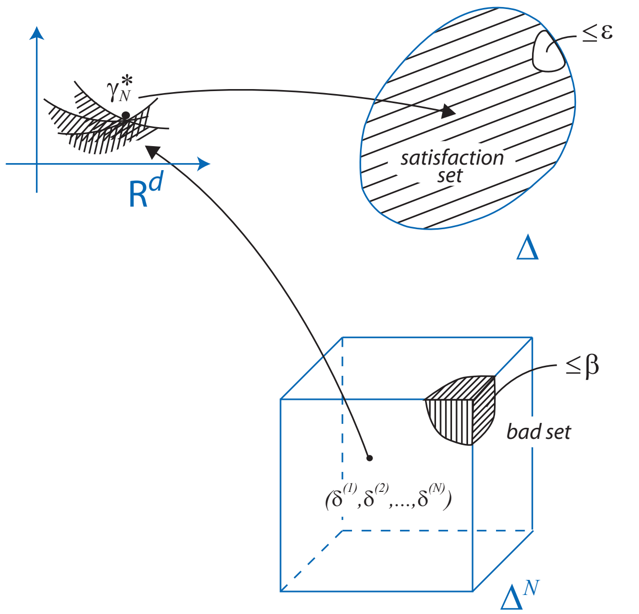

Fig. 1. A pictorial representation of Theorem 1.

In the figure, the N samples $\delta^{(1)}$, ..., $\delta^{(N)}$ extracted from Δ are represented as a single multi-extraction ($\delta^{(1)}$, ..., $\delta^{(N)}$) from Δ^N. In Δ^N a 'bad set' exists: if we extract a multi-sample in the bad set, no conclusions are drawn. But this has a very tiny probability to occur, 10^{-10} or 10^{-20}.

In all other cases, the multi-sample maps into a finite convex optimization problem that we can easily solve. The corresponding solution automatically satisfies all the other unseen constraints except for a small fraction ϵ. Thus, the scenario approach is a viable method to robustify any nominal design up to a level ϵ.

Theorem 1 is a *generalization* theorem in that it shows that the solution γ_N^* obtained by looking at a finite number of constraints generalizes to cope with unseen constraints. Generalization always calls for some structure linking seen situations to unseen ones, and it is worth noticing that the only structure required in Theorem 1 is *convexity*. As a consequence, Theorem 1 applies to all convex problems (e.g. linear, quadratic or semi-definite involving LMIs) with no limitations and it can be used in the more diverse fields of systems and control theory.

3. APPLICATION TO SYSTEMS AND CONTROL PROBLEMS

The aim of this section is to show the versatility of the scenario approach by introducing some paradigms in systems and control where applying the scenario approach opens up new routes in problem solvability. For a more effective presentation and to help readability, the introduction of such paradigms is made through simple – yet not simplistic – examples.

3.1 Paradigm 1: robust control

Consider the following ARMA (Auto-Regressive Moving-Average) system

$$y_{t+1} = ay_t + bu_t + c_1 w_t + c_2 w_{t-1}, \qquad (6)$$

where u_t and y_t are input and output, and w_t is a $WN(0,1)$ (white noise with zero mean and unitary variance) disturbance; a, b, c_1, and c_2 are real parameters, with $|a| < 1$ (stability condition) and $b \neq 0$ (controllability condition), whose value is not precisely known.

We assume that w_t is measurable, and the objective is to

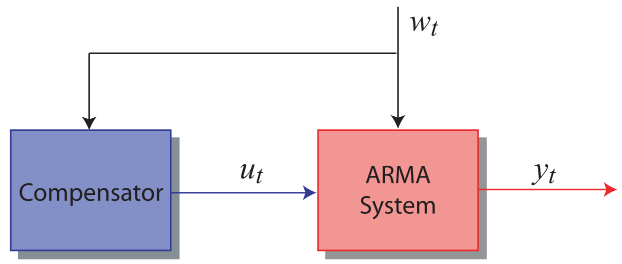

Fig. 2. The feed-forward compensation scheme.

design a feed-forward compensator with structure

$$u_t = k_1 w_t + k_2 w_{t-1}$$

that minimizes the asymptotic variance of y_t, see Figure 2.

If the system parameters a, b, c_1, and c_2 *were known*, an optimal compensator would be easily found. Indeed, substituting $u_t = k_1 w_t + k_2 w_{t-1}$ in (6) gives

$$y_{t+1} = ay_t + (c_1 + bk_1)w_t + (c_2 + bk_2)w_{t-1},$$

from which the expression for the asymptotic variance of y_t is computed as

$$E[y_t^2] = \frac{(c_1 + bk_1)^2 + (c_2 + bk_2)^2 + 2a(c_1 + bk_1)(c_2 + bk_2)}{1 - a^2}.$$

Hence, the values of k_1 and k_2 minimizing $E[y_t^2]$ are seen to be

$$k_1 = -\frac{c_1}{b} \quad \text{and} \quad k_2 = -\frac{c_2}{b}, \tag{7}$$

resulting in $E[y_t^2] = 0$.

On the other hand, the system parameter values are not always available in practical situations. More realistically, the parameters are only partially known, and they take value in a given uncertainty set Δ. In our example, this means that the compensator parameters k_1 and k_2 have to be designed according to some *robust* philosophy, e.g. the min-max approach:

$$\min_{k_1,k_2} \max_{a,b,c_1,c_2 \in \Delta} E[y_t^2] = \min_{k_1,k_2} \max_{a,b,c_1,c_2 \in \Delta} \ell_{(a,b,c_1,c_2)}(k_1, k_2), \tag{8}$$

where

$$\ell_{(a,b,c_1,c_2)}(k_1, k_2) = \frac{(c_1 + bk_1)^2 + (c_2 + bk_2)^2 + 2a(c_1 + bk_1)(c_2 + bk_2)}{1 - a^2}$$

(compare with (1)). For any value of a, b, c_1, c_2 with $|a| < 1$ and $b \neq 0$, function $\ell_{(a,b,c_1,c_2)}(k_1, k_2)$ is convex in k_1, k_2 (actually, it is a paraboloid).

The problem with solving (8) is that the uncertainty set Δ where the system parameters a, b, c_1, c_2 range depends on the particular problem at hand and can be complicated. In general, problem (8) cannot be solved analytically, and even standard numerical methods may fail to provide a solution.
In this case, the scenario approach represents a viable way to find an approximate solution to (8) with guaranteed performance.

As an example, suppose that the uncertainty set Δ is parameterized by $(\theta_1, \theta_2) \in [-1/3, 1/3]^2$ as follows:

$$\begin{aligned}\Delta = \{a, b, c_1, c_2 :\ & a = 0.45 + 0.5 \cdot (1 - \mathrm{e}^{-8\cdot 10^3(\theta_1^2+\theta_2^2)}),\\ & b = 1 + \theta_2^2,\\ & c_1 = 0.2 + (\theta_2 + \sin(\theta_2) + 0.1) \cdot \sin(2\pi\theta_2),\\ & c_2 = 0.5 + \theta_1^2 \cos(\theta_2),\\ & (\theta_1, \theta_2) \in [-1/3, 1/3]^2\}.\end{aligned}$$

The nominal values for θ_1 and θ_2 are $\theta_1^{nom} = 0$ and $\theta_2^{nom} = 0$ corresponding to $a^{nom} = 0.45$, $b^{nom} = 1$, $c_1^{nom} = 0.2$, and $c_2^{nom} = 0.5$.

According to the scenario approach with $\epsilon = 0.01$ and $\beta = 10^{-10}$, we extracted $N = 5206$ values of a, b, c_1 and c_2 from Δ (say $a^{(i)}$, $b^{(i)}$, $c_1^{(i)}$ and $c_2^{(i)}$, $i = 1, \dots, 5206$) by uniformly sampling N values for θ_1 and θ_2 from $[-1/3, 1/3]^2$.
The resulting scenario optimization problem with 5206 constraints is:

$$\min_{k_1,k_2,h} h$$
$$\text{subject to: } \ell_{(a^{(i)},b^{(i)},c_1^{(i)},c_2^{(i)})}(k_1, k_2) \leq h, \quad i = 1, \dots, 5206.$$

This problem has a linear objective and quadratic constraints, and was easily solved by the CVX solver for Matlab, Grant et al. [2006, 2007]. We obtained $k_1^* = -0.50$, $k_2^* = -0.53$ and $h^* = 1.16$.

According to Theorem 1, with probability $1-\beta = 1-10^{-10}$ (in practice with probability 1) the compensator $u_t = k_1^* w_t + k_2^* w_{t-1}$ guarantees $E[y_t^2] = \ell_{(a,b,c_1,c_2)}(k_1^*, k_2^*) \leq h^* = 1.16$ for all plants in the uncertainty set Δ but a small fraction of size at most $\epsilon = 0.01$.
Evidence of this robustness property can be found in Figure 3, where we plotted $\ell_{(a,b,c_1,c_2)}(k_1^*, k_2^*)$ as a function of the re-parametrization θ_1, θ_2 of a, b, c_1, and c_2. The flat surface is at the value $h^* = 1.16$.

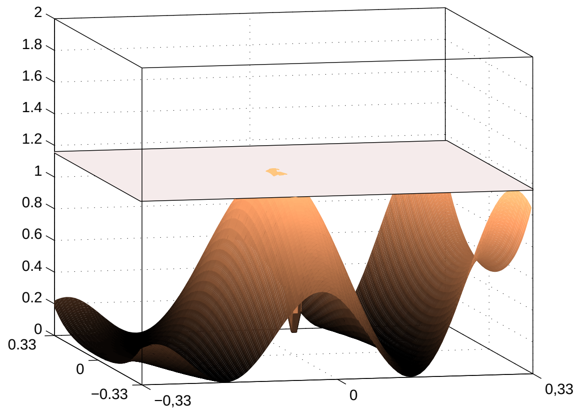

Fig. 3. $\ell_{(a,b,c_1,c_2)}(k_1^*, k_2^*)$ as a function of θ_1 and θ_2.

We also compared the robust compensator k_1^*, k_2^* with the nominal one $k_1^{nom} = -0.2$, $k_2^{nom} = -0.5$ (i.e. the optimal compensator as in (7) for the nominal system $a^{nom} = 0.45$, $b^{nom} = 1$, $c_1^{nom} = 0.2$, $c_2^{nom} = 0.5$).
Figure 4 depicts the output obtained when both compensators k_1^*, k_2^* and k_1^{nom}, k_2^{nom} were applied to the nominal system.

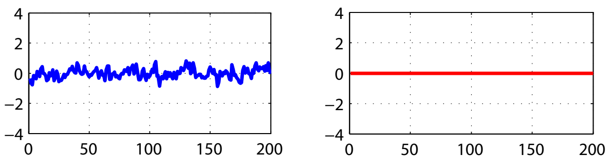

Fig. 4. Output of the nominal system with robust compensator k_1^*, k_2^* (left), and with nominal compensator k_1^{nom}, k_2^{nom} (right).

Not surprisingly, the performance of compensator k_1^*, k_2^* applied to the nominal system is worse than that of compensator k_1^{nom}, k_2^{nom}, being the latter optimal in this case. Yet, noise rejection remains quite good for k_1^*, k_2^*.

When we consider other plants in the uncertainty set, the performance of the nominal compensator gets worse than the safe-guard level h^* attained by the robust compensator k_1^*, k_2^*.

This is e.g. the case of Figure 5 where the system obtained by setting $\theta_1 = -0.21$, $\theta_2 = -0.32$ was considered. This shows the robustness features of the scenario design.

The applicability of the scenario methodology to robust control goes far beyond the simple noise rejection problem here considered, and, indeed, scenario design can be applied to many other paradigms in robust control such as robust stabilization, robust H_2 design, LPV control, robust

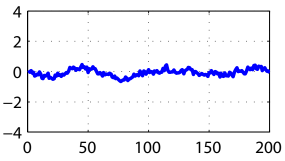

Fig. 5. Output of the system $a = 0.33$, $b = 1.04$, $c_1 = 0.60$, and $c_2 = 0.10$ with robust compensator k_1^*, k_2^* (left), and with nominal compensator k_1^{nom}, k_2^{nom} (right).

pole assignment, etc. See Calafiore and Campi [2006] for a sample of application examples.

3.2 Paradigm 2: control with saturation constraints

Consider a discrete time linear system with scalar input u_t and scalar output y_t, described by the following equation:

$$y_t = G(z)u_t + d_t,$$

where $G(z)$ is a known stable transfer function and d_t is an additive stochastic disturbance.

Denote by $\mathcal{D}$ the set of possible realizations of the disturbance d_t. Our objective is to design a feedback controller

$$u_t = C(z)y_t,$$

such that the disturbance is optimally attenuated for every realization in $\mathcal{D}$, while avoiding saturation of the control input due to actuator limitations.

The effect of the disturbance d_t is quantified through the finite-horizon 2-norm $\sum_{t=1}^{M} y_t^2$ of the closed-loop system output and the goal is choosing $C(z)$ which minimizes the worst-case disturbance effect

$$\max_{d_t \in \mathcal{D}} \sum_{t=1}^{M} y_t^2, \tag{9}$$

while maintaining the control input u_t within a saturation limit u_{sat}:

$$|u_t| \leq u_{\text{sat}},\ t = 1, 2, \ldots, M,\ \forall d_t \in \mathcal{D}. \tag{10}$$

Note that here uncertainty is not on the parameters of the plant and, indeed, $G(z)$ in this application is known; what is uncertain is the disturbance realization and the design has to be made so that attenuation is achieved robustly with respect to the disturbance realization.

This constrained optimization problem is now re-formulated as a robust convex optimization program by adopting the following Internal Model Control (IMC) parametrization (see Morari and Zafiriou [1989])

$$C(z) = \frac{Q(z)}{1 + G(z)Q(z)}$$

of the closed-loop stabilizing controllers, where $G(z)$ is the system transfer function and $Q(z)$ is a free-to-choose stable transfer function (see Figure 6). The IMC parametrization of the controller is particularly convenient since the transfer functions from d_t to u_t and from d_t to y_t are affine in $Q(z)$:

$$u_t = Q(z)d_t \tag{11}$$

$$y_t = [1 + G(z)Q(z)]d_t. \tag{12}$$

Consequently, if $Q(z)$ is linearly parameterized, e.g. it is the multi-lagged structure

$$Q(z) = q_0 + q_1 z^{-1} + q_2 z^{-2} + \cdots + q_k z^{-k}, \tag{13}$$

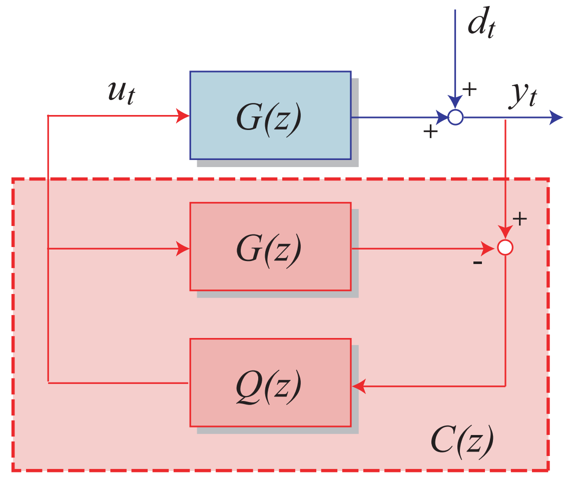

Fig. 6. The IMC parameterization.

the cost (9) and the constraints (10) are convex in $q := [q_0\, q_1\, \ldots\, q_k]^T \in \mathbb{R}^{k+1}$.

The control design problem can now be precisely formulated as the following robust convex optimization program:

$$\min_{q, h \in \mathbb{R}^{k+2}} h \tag{14}$$

$$\text{subject to: } \sum_{t=1}^{M} y_t^2 \leq h,\ \forall d_t \in \mathcal{D}, \tag{15}$$

$$|u_t| \leq u_{\text{sat}},\ t = 1, 2, \ldots, M,\ \forall d_t \in \mathcal{D}, \tag{16}$$

where the slack variable h represents an upper bound to the output 2-norm $\sum_{t=1}^{M} y_t^2$ for any realization of d_t (see (15)). Such an upper bound is minimized in (14) under the additional constraint (16) that u_t does not exceed the saturation limits.

The constraints (15) and (16) can be made more explicit as a function of q. For example, when $Q(z)$ is given by (13), problem (14)-(16) is equivalent to

$$\begin{aligned} &\min_{q, h \in \mathbb{R}^{k+2}} h \\ &\text{subject to: } \quad q^T A q + Bq + C \leq h,\ \forall d_t \in \mathcal{D} \\ &\qquad\qquad\qquad |\phi_t^T q| \leq u_{\text{sat}},\ t = 1, 2, \ldots, M,\ \forall d_t \in \mathcal{D}, \end{aligned}$$

where A, B, C, and ϕ_t are suitable matrices that depend on d_t. Indeed, by (11), (12), and the parametrization of $Q(z)$ in (13), the input and the output of the controlled system can be expressed as

$$u_t = \phi_t^T q$$

$$y_t = \psi_t^T q + d_t,$$

where ϕ_t and ψ_t are vectors containing delayed and filtered versions of the disturbance d_t:

$$\phi_t = \begin{bmatrix} d_t \\ d_{t-1} \\ \vdots \\ d_{t-k} \end{bmatrix} \text{ and } \psi_t = \begin{bmatrix} G(z)d_t \\ G(z)d_{t-1} \\ \vdots \\ G(z)d_{t-k} \end{bmatrix}. \tag{17}$$

Then, $\sum_{t=1}^{M} y_t^2$ can be expressed as $\sum_{t=1}^{M} y_t^2 = q^T A q + Bq + C$, where

$$A = \sum_{t=1}^{M} \psi_t \psi_t^T,\ B = 2\sum_{t=1}^{M} d_t \psi_t^T,\ C = \sum_{t=1}^{M} d_t^2 \tag{18}$$

are matrices that depend on d_t.

The implementation of the scenario optimization in this control set-up requires to randomly extract a certain number N of disturbance realizations $d_t^{(1)}, \ldots, d_t^{(N)}$ and to compute ϕ_t and ψ_t in (17) for the extracted $d_t^{(i)}$ by simulating on a computer the output of system $G(z)$ fed by $d_t^{(i)}$ (for this reason we refer to this approach as 'by simulation'). This leads to the following scenario optimization program:

$$\min_{q,h \in \mathbb{R}^{k+2}} h \tag{19}$$

$$\text{subject to: } q^T A^{(i)} q + B^{(i)} q + C^{(i)} \leq h, \; i = 1, \ldots, N,$$

$$|\phi_t^{(i)T} q| \leq u_{\rm sat}, \; t = 1, 2, \ldots, M, i = 1, \ldots, N,$$

where $\phi_t^{(i)}$ and $A^{(i)}$, $B^{(i)}$, $C^{(i)}$ are as in (17) and (18) with $d_t = d_t^{(i)}$.

Before presenting numerical results, it is perhaps of interest to remark the fact that in the scenario approach the uncertain element δ is totally general and can e.g. be the plant parameters as in the previous section or a disturbance realization as in the present context.

We next present numerical results when $G(z) = \frac{1}{z-0.8}$, and the stochastic disturbance d_t is sinusoidal with frequency $\frac{\pi}{8}$, i.e.

$$d_t = \alpha_1 \sin(\frac{\pi}{8} t) + \alpha_2 \cos(\frac{\pi}{8} t),$$

where α_1 and α_2 are independent random variables uniformly distributed in $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

As for the IMC parametrization $Q(z)$ in (13), we choose $k = 1$ and $Q(z) = q_0 + q_1 z^{-1}$.

A control design problem (14)–(16) is considered with $M = 300$, and for three different values of the saturation limit $u_{\rm sat}$: 2, 0.8, and 0.2.

In the scenario approach, we let $\epsilon = 0.05$ and $\beta = 10^{-10}$. Correspondingly, N given by (4) is $N = 1042$. Let (q^*, h^*) be the solution to (19) with $N = 1042$. Then, with probability no smaller than $1 - 10^{-10}$, the designed controller with parameter q^* guarantees the upper bound h^* on the output 2-norm $\sum_{t=1}^{300} y_t^2$ over all disturbance realizations, except for a fraction of them whose probability is smaller than or equal to 0.05. At the same time, the control input u_t is guaranteed not to exceed the saturation limit $u_{\rm sat}$ except for the same fraction of disturbance realizations.

Figures 7, 8, and 9 represent the pole-zero maps and the Bode plots of the transfer function $F(z) = 1 + G(z)Q(z)$ between the disturbance d_t and the controlled output y_t (sensitivity function), for decreasing values of $u_{\rm sat}$ (2, 0.8, 0.2).
When the saturation bound is large ($u_{\rm sat} = 2$), the outcome of the design is a controller that efficiently attenuates the sinusoidal disturbance at frequency $\frac{\pi}{8}$ by placing a pair of zeros approximately in $e^{\pm i\pi/8}$ in the sensitivity transfer function. As $u_{\rm sat}$ decreases, the control effort required to neutralize the sinusoidal disturbance exceeds the saturation constraint, and a design with damped zeros is automatically chosen.
The values of the cost $\max_{d_t^{(i)}, i=1,2,\ldots,N} \sum_{t=1}^{300} y_t^2 = h^*$ for

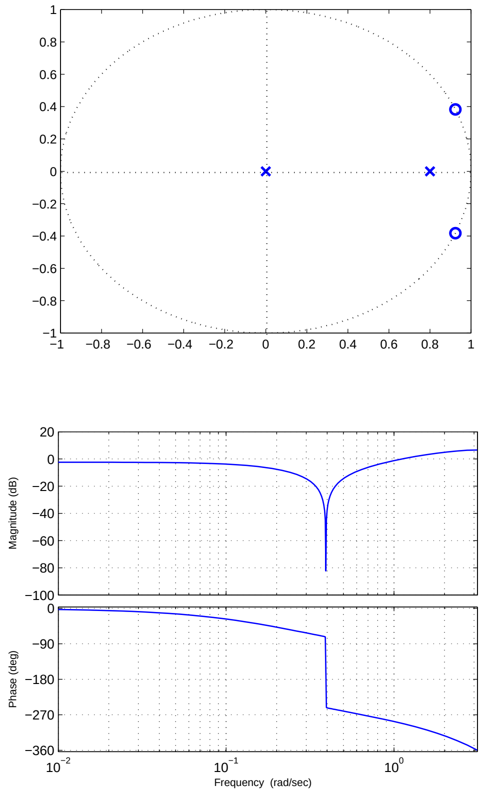

Fig. 7. Pole-zero map and Bode plot of the sensitivity function for $u_{\rm sat}$ =2.

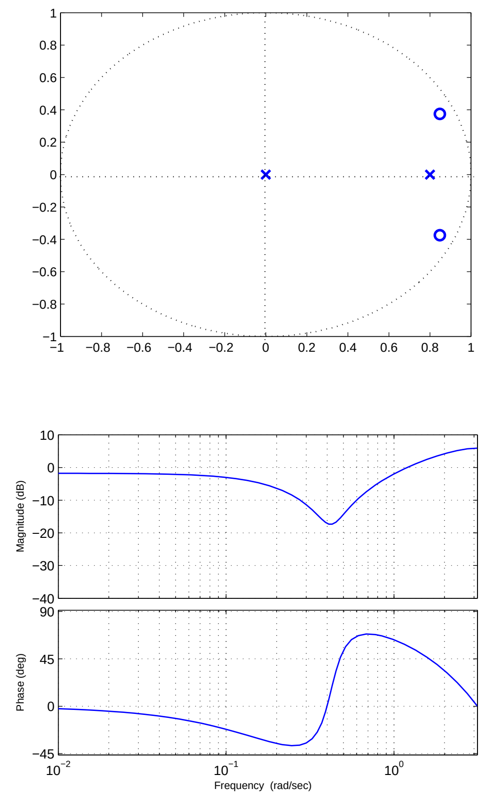

Fig. 8. Pole-zero map and Bode plot of the sensitivity function for $u_{\rm sat}$ =0.8.

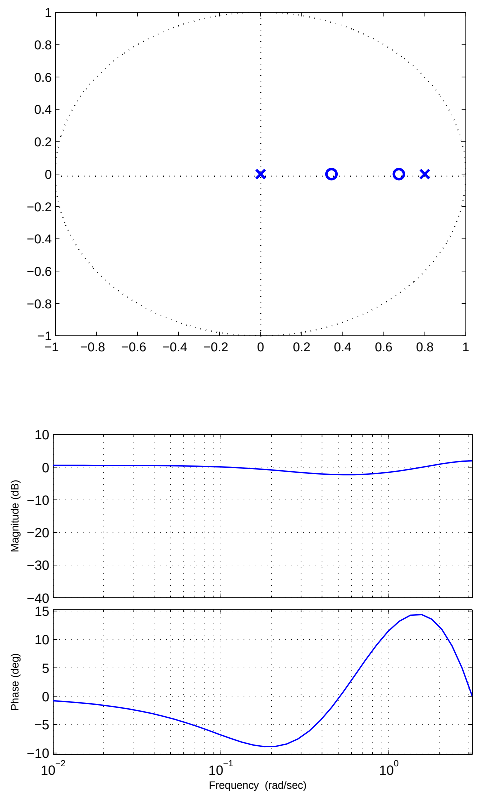

Fig. 9. Pole-zero map and Bode plot of the sensitivity function for u_{sat} =0.2.

$u_{\text{sat}} = 2$, 0.8, and 0.2, are respectively equal to 0.75, 3.61, and 90.94. As expected, h^* increases as u_{sat} decreases, since the saturation constraint on u_t becomes progressively more stringent.
Note that, when the saturation bound is equal to 2, the scenario solution coincides with the solution that one would naturally conceive without taking into account the saturation constraint. However, when the saturation constraint becomes more stringent, the design is more tricky.

Before closing this section, it is perhaps worth noticing that the paradigm of disturbance rejection with limitations on the control action here developed can be easily extended to more general control settings including reference tracking with constraints of different kinds.

3.3 Paradigm 3: model reduction

In many application contexts, a model of the system under consideration is available for simulation purposes (simulator). In most cases this simulator is derived from first principles and with the ideal objective of resembling the system behavior in all operating conditions, which may result in a complex nonlinear model of high dimension and, possibly, described through PDEs.
Due to its intrinsic complexity, using the simulation model in design problems is difficult, if not impossible. A typical way around this problem is to first derive a simpler low-dimensional model that best fits the system behavior in the operating conditions of interest, and then perform the design based on this model. The performance of the obtained design can be eventually verified on the simulator prior to implementation.

The term 'model reduction' refers to the vast area of systems theory that studies the problem of deriving a 'reduced' model of a system. Normally, the model reduction problem is tackled by examining the structure of the system and by trying to simplify such a structure so as to also preserve some relevant characteristics of the initial system.

An alternative way to go consists in running a set of experiments and in measuring the system response to some input signals of interest. A reduced model of predefined structure is then tuned so as to resemble the observed system behavior. When a simulator of the system is available, this approach to model reduction becomes particularly attractive since one can run a number of experiments on the simulator rather than on the real system. An important point we want to make here is that the scenario approach allows to assess how many experiments are needed to obtain a reduced model with guaranteed performance, and that this number does not depend on the system complexity but only on the complexity of the model to be tuned, see also Bittanti et al. [2007, 2009] for more comments on this point.

More formally, given a simulator $\mathcal{S}$ and a class of models parameterized by $\theta \in \mathbb{R}^k$, suppose that the accuracy of model $\mathcal{M}_\theta$ in reproducing the output of $\mathcal{S}$ when fed by the input signal u_t is quantified by a cost function $J_{u_t}(\theta)$. For example, $J_{u_t}(\theta)$ can be taken as the 2-norm of the error signal $\mathcal{S}[u_t] - \mathcal{M}_\theta[u_t]$ between the output $\mathcal{S}[u_t]$ of the simulator and the output $\mathcal{M}_\theta[u_t]$ of the model with parameter θ: $J_{u_t}(\theta) = \|\mathcal{S}[u_t] - \mathcal{M}_\theta[u_t]\|_2$. Then, the worst-case accuracy achieved by $\mathcal{M}_\theta$ over the set U of input signals u_t of interest is given by

$$\max_{u_t \in U} J_{u_t}(\theta)$$

and the best model is $\mathcal{M}_{\theta^*}$, where θ^* is obtained by solving the min-max optimization problem:

$$\min_{\theta} \max_{u_t \in U} J_{u_t}(\theta). \tag{20}$$

As discussed in Section 1, the min-max problem (20) can be rewritten as the robust optimization problem:

$$\min_{\theta, h \in \mathbb{R}^{k+1}} h \tag{21}$$
$$\text{subject to: } J_{u_t}(\theta) \leq h, \ \forall u_t \in U,$$

with u_t representing the uncertainty parameter taking value in the possibly infinite uncertainty set U.

If the cost $J_{u_t}(\theta)$ is convex as a function of θ (this is, e.g, the case when $\mathcal{M}_\theta$ is linearly parameterized in θ), then the scenario approach can be applied to (21). This involves extracting N input signals $u_t^{(i)}$, $i = 1, 2, \ldots, N$, from U, and running N experiments where in each experiment the simulator $\mathcal{S}$ is fed by input $u_t^{(i)}$, and output $\mathcal{S}[u_t^{(i)}]$ is measured. If N is chosen so as to satisfy (4) with $d = k + 1$ for some given ϵ and β, the obtained scenario solution (θ^*, h^*) is such that the reduced model $\mathcal{M}_{\theta^*}$ has guaranteed accuracy h^* over all input signals $u_t \in U$ except at most an ϵ-fraction, and this holds with probability at least $1 - \beta$. If the achieved accuracy level h^* is unsatisfactory, this is a sign that the reduced model class is too restricted and one can move to consider a more complex reduced model class.

It is important to note that the number N of experiments is determined independently of how complex the simulator is, and that this number depends only on the complexity of the reduced model to be designed, through the size k of its parametrization θ. This approach to model reduction actually does not require any knowledge on the structure of the simulator, since the simulator is only used to generate data.

4. CONCLUSIONS

In this paper, we provided an overview on the so-called scenario approach with specific focus on systems and control applications. The approach basically consists of the following main steps:

- reformulation of the problem as a robust (with infinite constraints) *convex* optimization problem;
- randomization over constraints and resolution (by means of standard numerical methods) of the so-obtained *finite* optimization problem;
- evaluation of the constraint satisfaction level of the obtained solution through Theorem 1.

The versatility of the scenario approach was illustrated through simple examples of systems and control design.

More details both on theoretical aspects and applications can be found in the technical literature.
In particular, the theory of the scenario approach has been developed in the last four years in Calafiore and Campi [2005, 2006], Campi and Garatti [2008], while it has also been extended to design with robustness modulation in Campi and Garatti [2007].
As for applications, robust control is treated in Calafiore and Campi [2003b, 2004, 2006], with reference among others to robust stabilization, robust H_2 design, LPV (Linear Parameter Varying) control, and robust pole assignment. The main references for control by simulation are Prandini and Campi [2007, 2009], while model reduction is a new application framework currently underway, here presented for the first time.
It is perhaps worth mentioning that another setting in the systems and control area where the scenario approach proved powerful (and which was not illustrated in this paper since it would have led us too far afield) is the identification of interval predictor models, i.e. models returning a prediction interval instead of a single prediction value. The main references are Calafiore and Campi [2003a], Campi et al. [2009], Garatti and Campi [2009].

REFERENCES

T. Alamo, R. Tempo, and E.F. Camacho. Improved samples size bounds for robust control design: A pack-based strategy. In *Proceedings of the* 46^{th} *IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007.

T. Alamo, R. Tempo, and E.F. Camacho. Statistical learning theory: t pack-based strategy for uncertain feasibility and optimization problems. In V.D. Blondel, S.P. Boyd, and H. Kimura, editors, *Recent Advances in Learning and Control*, Springer- Verlag, London, 2008.

P. Apkarian and H.D. Tuan. Parameterized LMIs in control theory. *SIAM Journal on Control and Optimization*, 38(4):1241–1264, 2000.

P. Apkarian, H.D. Tuan, and J. Bernussou. Continuous-time analysis, eigenstructure assignment, and H_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations. *IEEE Transactions on Automatic Control*, 46(12):1941–1946, 2001.

B.R. Barmish and C.M. Lagoa. The uniform distribution: A rigorous justification for its use in robustness analysis. *Math. Control, Signals, Syst.*, 10:203–222, 1997.

A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1):1–13, 1999.

A. Ben-Tal and A. Nemirovski. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM Journal on Optimization*, 12 (3):811–833, 2002.

S. Bittanti, M.C. Campi, and M. Prandini. How many experiments are needed to adapt? In A. Chiuso, A. Ferrante, and S. Pinzoni, editors, *Modeling, Estimation and Control. Festschrift in Honor of Giorgio Picci on the Occasion of his Sixty-Fifth Birthday*, pages 7-17, Springer- Verlag, 2007.

S. Bittanti, M.C. Campi, and M. Prandini. Adaptation and the effort needed to adapt. In *Proceedings of the* 48^{th} *IEEE Conference on Decision and Control*, Shanghai, China, to appear.

V.D. Blondel and J.N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36:1249–1274, 2000.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.

S. Boyd and V. Vandenberghe. *Convex optimization*. Cambridge University press, Cambridge, UK, 2004.

R.P. Braatz, P.M. Young, J.C. Doyle, and M. Morari. Computational complexity of μ calculation. *IEEE Transactions on Automatic Control*, AC-39:1000–1002, 1994.

G. Calafiore. A note on the expected probability of constraint violation in sampled convex programs. *Available online at http://www.optimization-online.org/DB_HTML/2008/07/2046.html*, 2008.

G. Calafiore and M.C. Campi. A learning theory approach to the construction of predictor models. In *Discrete and Continuous Dynamical Systems, supplement volume*, pages 156–166, 2003a.

G. Calafiore and M.C. Campi. Robust convex programs: randomized solutions and application in control. In *Proceedings of the* 42^{nd} *IEEE Conference on Decision and Control*, Maui, Hawaii, USA, 2003b.

G. Calafiore and M.C. Campi. A new bound on the generalization rate of sampled convex programs. In *Proceedings of the* 43^{rd} *IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, USA, 2004.

G. Calafiore and M.C. Campi. Uncertain convex programs: randomized solutions and confidence levels. *Mathemat-*

ical Programming, 102(1):25–46, 2005.

G. Calafiore and M.C. Campi. The scenario approach to robust control design. *IEEE Transactions on Automatic Control*, 51(5):742–753, 2006.

G. Calafiore, F. Dabbene, and R. Tempo. Randomized algorithms for probabilistic robustness with real and complex structured uncertainty. *IEEE Transactions on Automatic Control*, 45(12):2218–2235, 2000.

M.C. Campi and S. Garatti. Modulating robustness in robust control: making it easy through randomization. In *Proceedings of the* 46^{th} *IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007.

M.C. Campi and S. Garatti. The exact feasibility of randomized solutions of robust convex programs. *SIAM Journal on Control and Optimization*, 19(3):1211–1230, 2008.

M.C. Campi, G. Calafiore, and S. Garatti. Interval predictor models: identification and reliability. *Automatica*, 45:382–392, 2009.

Y. Fujisaki, F. Dabbene, and R. Tempo. Probabilistic robust design of LPV control systems. *Automatica*, 39: 1323-1337, 2003.

P. Gahinet. Explicit controller formulas for LMI-based H_∞ synthesis. *Automatica*, 32(7):1007–1014, 1996.

L. El Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM Journal on Matrix Analysis and Applications*, 18(4):1035–1064, 1997.

L. El Ghaoui and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization*, 9(1):33–52, 1998.

G.S. Fishman. *Monte Carlo: concepts, algorithms and applications*. Springer-Verlag, New York, 1999.

S. Garatti and M.C. Campi. L-infinity layers and the probability of false prediction. In *Proceedings of the IFAC Symposium on System Identification, SYSID*, Saint Malo, 2009.

G.C. Goodwin, M.M. Seron, and J.A. De Doná. *Constrained control and estimation: an optimisation approach*. Springer-Verlag, New York, NY, USA, 2005.

M. Grant, S. Boyd, and Y. Ye. Disciplined convex programming. In L. Liberti and N. Maculan, editors, *Global Optimization: from Theory to Implementation, Nonconvex optimization and its applications*, pages 155–210, New York, NY, USA, 2006. Springer Science. *Available online at http://www.stanford.edu/∼boyd/disc_cvx_prog.html*.

M. Grant, S. Boyd, and Y. Ye. Cvx: Matlab software for disciplined convex programming. *Available online at http://www.stanford.edu/∼boyd/cvx/*, 2007.

J.M. Hammersley, and D.C. Handscomb. *Monte Carlo methods*. Methuen & Co Ltd, London, 1964.

S. Kanev, B. De Schutter, and M. Verhaegen. An ellipsoid algorithm for probabilistic robust controller design. *Systems & Control Letters*, 49:365-375, 2003.

P.P. Khargonekar and A. Tikku. Randomized algorithms for robust control analysis have polynomial time complexity. In *Proceedings of the* 35^{th} *IEEE Conference on Decision and Control*, Kobe, Japan, 1996.

C.M. Lagoa. Probabilistic enhancement of classical robustness margins: a class of nonsymmetric distributions. *IEEE Transactions on Automatic Control*, 48(11):1990–1994, 2003.

C.M. Lagoa, P.S. Shcherbakov, and B.R. Barmish. Probabilistic enhancement of classical robustness margins: The unirectangularity concept. *Systems & Control Letters*, 35:31–43, 1998.

M. Morari and E. Zafiriou. *Robust process control*. Prentice Hall, Englewood Cliffs, NJ, USA, 1989.

A. Nemirovski. Several NP-hard problems arising in robust stability analysis. *SIAM Journal on Matrix Analysis and Applications*, 6:99–105, 1993.

Y. Oishi and H. Kimura. Computational complexity of randomized algorithms for solving parameter-dependent linear matrix inequalities. *Automatica*, 39(7):2149-2156, 2003.

B.T. Polyak and R. Tempo. Probabilistic robust design with linear quadratic regulators. *Systems & Control Letters*, 43:343-353, 2001.

M. Prandini and M.C. Campi. A new approach to controller design in presence of constraints. In *Proceedings of the* 46^{th} *IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007.

M. Prandini and M.C. Campi. Scenario-based control: a new approach to constrained control design. Technical report, University of Brescia, 2009.

L.R. Ray and R.F. Stengel. A Monte Carlo approach to the analysis of control system robustness. *Automatica*, 29:229–236, 1993.

C.P. Robert and G. Casella. *Monte Carlo statistical methods*. Springer, 2004.

C.W. Scherer. Relaxations for robust linear matrix inequality problems with verifications for exactness. *SIAM Journal on Matrix Analysis and Applications*, 27(2): 365–395, 2005.

C.W. Scherer. LMI relaxations in robust control. *European Journal of Control*, 12:3–29, 2006.

A. Shapiro, D. Dentcheva, and A. Ruszczynski. *Lectures on stochastic programming: modeling and theory*. SIAM, Philadelphia, 2009, to appear.

R.F. Stengel and L.R. Ray. Stochastic robustness of linear time-invariant control systems. *IEEE Transactions on Automatic Control*, 36:82–87, 1991.

R. Tempo, E.W. Bai, and F. Dabbene. Probabilistic robustness analysis: explicit bounds for the minimum number of samples. *Systems & Control Letters*, 30:237–242, 1997.

R. Tempo, G. Calafiore, and F. Dabbene. *Randomized algorithms for analysis and control of uncertain systems*. Springer-Verlag, London, UK, 2005.

L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, 1996.

M. Vidyasagar. *A Theory of Learning and Generalization*. Springer-Verlag, London, UK, 1997.

M. Vidyasagar. Randomized algorithms for robust controller synthesis using statistical learning theory. *Automatica*, 37(10):1515–1528, 2001.